Totally inverse inequality
https://www.linkedin.com/feed/update/urn:li:activity:6556025381873430529
Let a, b, c are positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{a+b+c} \geq 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \cdot \frac{1}{a^2+b^2+c^2}.$$

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Let $s := a+b+c, p := ab+bc+ca$. Then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc} = p$,
$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{a^2b^2+b^2c^2+c^2a^2}{a^2b^2c^2} = p^2 - 2s, a^2+b^2+c^2 = s^2 - 2p$
and inequality of the problem becomes

(1) $\quad p - \frac{3}{s} \geq \frac{2(p^2-2s)}{s^2-2p}.$

Since $s^2 = (a+b+c)^2 \geq 3(ab+bc+ca) = 3p$ then $p^2 - 2s > 0$ and
(1)$\Leftrightarrow (sp-3)(s^2-2p) \geq 2s(p^2-2s) \Leftrightarrow -4p^2s + p(s^3+6) + s^2 \geq 0.$
We have $s \geq 3(abc)^{1/3} = 3$ and since $9 \geq 4sp - s^3$ (Schure's Inequality $\sum a(a-b)(a-c) \geq 0$ in s,p-notation and normalized by $abc = 1$) then

$$\sqrt{3s} \leq p \leq \frac{9+s^3}{4s}, s \geq 3.$$

For quadratic function $h(p) := -4p^2s + p(s^3+6) + s^2$ and $p \in \left[\sqrt{3s}, \frac{9+s^3}{4s}\right]$
we have $\min h(p) = \min\left\{h\left(\sqrt{3s}\right), h\left(\frac{9+s^3}{4s}\right)\right\} \geq 0$ because

$$h\left(\frac{9+s^3}{4s}\right) = -4\left(\frac{9+s^3}{4s}\right)^2 s + \frac{9+s^3}{4s}(s^3+6) + s^2 = \frac{(s-3)(s^2+3s+9)}{4s} \geq 0$$

and $h\left(\sqrt{3s}\right) = -4 \cdot 3s \cdot s + \sqrt{3s}\,(s^3+6) + s^2 =$

$$\frac{\left(s\sqrt{3s} - 2\right)\left(\sqrt{s} - \sqrt{3}\right)\left(s\sqrt{3s} + 3s + 3\sqrt{3s}\right)}{\sqrt{3}} \geq 0.$$